HALF-LIFE EQUATIONS

The basic equation…the starting point…:

$$y = \frac{1}{2^x} \quad \text{written for time:} \quad y = \frac{1}{2^{t_{1/2}}}$$

where y = fraction of original material
and $t_{1/2}$ = number of half-lives
∴,

$$2^{t_{1/2}} = \frac{1}{y}$$

and

$$t_{1/2} = \log_2\left(\frac{1}{y}\right)$$

to calculate the age (# years):

$$t_{age} = (\text{half-life}) * t_{1/2}$$

$$\mathbf{t_{age} = (half\text{-}life) * \log_2\left(\frac{1}{y}\right)}$$

That's really all there is to it….The equations really are that simple!

The following pages have examples and explanations of how this simple form of the equation is the same as the equations in the Historical Geology Lab Manual…and the same as equation found in textbooks.

$$t_{age} = (\text{half-life}) * \log_2\left(\frac{1}{y}\right) \quad = \quad t_{age} = \left(\frac{\text{halflife}}{0.693}\right) * \ln\left(1/y\right) \quad = t_{age} = \frac{(-1)}{K} * \ln\left(\frac{n_o}{n_t}\right)$$

$$\text{….and } t_{age} = \frac{(-1)}{K} * \ln\left(\frac{n_o}{n_t}\right) = t_{age} = \left(\frac{1}{K}\right) * \ln\left(\frac{n_t}{n_o}\right); \text{ since } \log\left(\frac{1}{x}\right) = -\log(x) \text{ ……}$$

$$K = \frac{0.693}{\text{halflife}} = \frac{\ln(2)}{\text{halflife}}$$

and $y = \left(\frac{n_o}{n_t}\right)$

y = fraction of original material
n_o = amount of parent material left
n_t = total amount of material = parent+daughter

The foundation…the basics…where it all begins…..

$$y = \frac{1}{2^{t_{1/2}}}$$

where y = fraction of original material
and $t_{1/2}$ = number of half-lives

Examples….

How much is left after 1 half-life?

$$y = \frac{1}{2^1} = \frac{1}{2}$$

How much is left after 2 half-lives?

$$y = \frac{1}{2^2} = \frac{1}{4}$$

How much is left after 3 half-lives?

$$y = \frac{1}{2^3} = \frac{1}{8}$$

How much is left after 4 half-lives?

$$y = \frac{1}{2^4} = \frac{1}{16}$$

How much is left after 5 half-lives?

$$y = \frac{1}{2^5} = \frac{1}{32}$$

$$y = \frac{1}{2^{t_{1/2}}}$$ ☺

BASIC LOGARITHMS

Example…start here:

$2^1 = 2$
$2^2 = 2*2 = 4$
$2^3 = 2*2*2 = 8$
$2^4 = 2*2*2*2 = 16$

Logarithm notation reads as follows:

	In logarithm notation
$2^1 = 2$	$\log_2(2)= 1$
$2^2 = 4$	$\log_2(4)= 2$
$2^3 = 8$	$\log_2(8)= 3$
$2^4 = 16$	$\log_2(16)= 4$

Take the last one…. $\log_2(16)= 4$
….This means: 2 must be raised to the 4th power for a final product of 16.

Try another one….

$\log_5(125)= ?$

….This means: what power of 5 yields a final product of 125?

….well… 5*5= 25….and 25*5=125……therefore….5*5*5=125... or 5^3=125…..

…therefore…. $\log_5(125)= 3$

From page 1…..

$$2^{t_{1/2}} = \frac{1}{y}$$

$$\log_2\left(\frac{1}{y}\right) = t_{1/2}$$

These are the basic half-life equations!
Its really that simple!

Most references present equations that have **ln** (natural logarithms) and do NOT have **log base 2…..** For example, here's one of these ln equations derived from one of the equations in the Historical Geology Lab Manual.

$$t_{age} = \left(\frac{\text{halflife}}{0.693}\right) * \ln\left(\frac{1}{y}\right)$$

This equation is actually the same equation as the one highlighted on the previous page….see below….

To convert from log base 2 to log base e (natural log):

Start with: $$2^{t_{1/2}} = \frac{1}{y} \quad \text{(from the first page)}$$

Take the natural log of both sides $$\ln\left(2^{t_{1/2}}\right) = \ln\left(\frac{1}{y}\right)$$

$$\therefore \quad t_{1/2} * \ln(2) = \ln\left(\frac{1}{y}\right)$$

$$t_{1/2} = \frac{\ln\left(1/y\right)}{\ln(2)}$$

and ln(2) = 0.693…. $$t_{1/2} = \frac{\ln\left(1/y\right)}{0.693}$$

to calculate the ages, multiply by the half-life

$$t_{age} = (\text{half-life}) * t_{1/2} \quad \ldots\ldots \rightarrow \quad t_{age} = (\text{half-life}) * \frac{\ln\left(1/y\right)}{0.693}$$

$$t_{age} = \left(\frac{\text{halflife}}{0.693}\right) * \ln\left(1/y\right)$$

Therefore…..

$$t_{age} = (\text{half-life}) * \log_2\left(\frac{1}{y}\right) \quad = \quad t_{age} = \left(\frac{\text{halflife}}{0.693}\right) * \ln\left(1/y\right)$$

Another version of the equation:

$$t_{age} = \frac{(-1)}{K} * \ln\left(\frac{n_o}{n_t}\right)$$

….which is the same as the 2 equations highlighted on the previous page…see below…

$$K = \frac{0.693}{\text{halflife}}$$

and $y = \left(\frac{n_o}{n_t}\right)$

y = fraction of original material
n_o = amount of parent material left
n_t = total amount of material left = parent+daughter

Starting with $t_{age} = \left(\frac{\text{halflife}}{0.693}\right) * \ln\left(\frac{1}{y}\right)$ from the previous page

$$t_{age} = \left(\frac{1}{K}\right) * \ln\left(\frac{1}{\left(n_o/n_t\right)}\right)$$

$$t_{age} = \left(\frac{1}{K}\right) * \ln\left(\frac{n_t}{n_o}\right)$$

…….and since $\log\left(\frac{1}{x}\right) = -\log(x)$ ……

$$t_{age} = \left(\frac{-1}{K}\right) * \ln\left(\frac{n_o}{n_t}\right)$$ (Robert's equation)

$\therefore t_{age} = \frac{(-1)}{K} * \ln\left(\frac{n_o}{n_t}\right)$ = $\mathbf{t_{age} = (\text{half-life}) * \log_2\left(\frac{1}{y}\right)}$ **the basic equation from the first page**

In conclusion, the basic equation for t_{age} (from the first page)…
is the same as the t_{age} equation in the Historical Geology Lab Manual and
is the same as the equation found in most textbooks

See the calculations on the next page for examples…..

		ln version of the Eqn		**Basic Eqn from page 1**	
4.50E+09	halflife	**K=(ln2)/(half-life)**		**$t=(\text{half-life})*\log_2(^1/_y)$**	
		$t=(-1)*(1/K)*\ln(n_o/n_t)$		**$t=(\text{half-life})*\log_2(n_t/n_o)$**	
Island	**n_o/n_t (y)**	**Calculated Age of Island**		**Calculated Age of Island**	
Hawaii	0.99992	**519,391**	years	519,391	years
Maui	0.99983	**1,103,756**	years	1,103,756	years
Molokai	0.99975	**1,623,235**	years	1,623,235	years
Oahu	0.99961	**2,532,424**	years	2,532,424	years
Kauai	0.99927	**4,740,984**	years	4,740,984	years
Niihau	0.99907	**6,040,488**	years	6,040,488	years
$y=1/2^t$					
$t=\log_2(^1/_y)$	where t is the number of half-lives….(not years)				
$\log(x) = (-1)*\log(^1/_x)$					
n_o = amount of parent material left					
n_t = total amount of material (parent+daughter)					
ln(2) = 0.693….when rounded….the calculations above use ln(2) before rounding					

More Basic Logarithms…..

TO TEST: $\boldsymbol{log_b(m^n) = n \cdot log_b(m)}$

$\log_3(81) = 4$

$\log_3(81) = \log_3(9)^2 = 2 \log_3(9) = 2*2 = 4$…..that works….

<u>First prove this:</u> $log_b(mn) = log_b(m) + log_b(n)$

$log_b(m) = Z$ and $log_b(n) = W$……..where Z and W are not known

$\therefore b^Z = m$ and $b^W = n$ and $b^Q = mn$

$\therefore log_b(mn) = log_b(b^Z\, b^W) = log_b(b^{Z+W}) = Z + W = log_b(m) + log_b(n)$

Then this almost prove itself!

$log_b(m^n) = n \cdot log_b(m)$

$log_b(m^n) = log_b(m*m*m*m$……. n times…..$) = log_b(m) + log_b(m) + log_b(m) +$ ….. n times

$\therefore log_b(m^n) = n \cdot log_b(m)$

The followin2 pages are from
http://physics.mtsu.edu/~wmr/log_2.htm
by Dr. William Robertson
Assistant Professor of Physics at Middle Tennessee State University

The Rules of Logarithms

Now that you understand what a logarithm means I want to show you a few simple mathematical manipulations that can be done using logarithms. These are often called the Rules of Logarithms, however they should not be mysterious to you given what we have covered in the previous sections. The first *rule* is typically expressed as

$$\log(XY) = \log X + \log Y$$

In words this rule states that if I take two numbers X and Y and multiply them together and then take the logarithm I obtain the same result as if I had added the logarithms of the two numbers separately. Consider the simple example of 10^2 times 10^3. Multiplied together these come to 10^5 and the log of 10^5 is 5 (that is what the left hand side of the equation tells us to do). Now the log of 10^2 is 2 and the right hand side of the equation instructs us to add this to the log of 10^3 which is 3 to get 5. The rule works! Of course this relation is nothing more than putting together the two ideas (1) that the log of a number is the exponent of 10 required to equal that number and (2) when we multiply powers of 10 we add the exponents.

The second rule of logarithms is a straightforward extension of the first--it states that

$$\log (X^n) = n \log X$$

that is, if the number X is raised to the power n the result is the same as n times log of X. To understand this rule imagine that n=2. In that case

$$\log (X^2) = \log (XX) = \log (X) + \log (X) = 2 \log X$$

where I merely wrote out X2 as XX in the first step and then used the first rule above. The extension to higher n should be obvious--try writing it out for n=3.

Finally there is a rule for division as well as for multiplication. This rule states that

$$\log(X/Y) = \log X - \log Y$$

Exponents and Logarithms

We have seen that when two numbers are multiplied we just add the corresponding exponents. You should also be getting a vague sense of how this relation is connected to the logarithmic nature of hearing described in the introductory module in that a multiplicative relation becomes an additive one. Now we want to introduce the logarithm. The concept of a logarithm is to merely replace a number by the exponent to which 10 would have to be raised to get that number. For example, consider the number 100. To what power would 10 have to be raised to get 100? I hope you cried out 2, since $10^2 = 100$. Thus the logarithm of 100 is 2. In mathematical terms we would write this as log(100) = 2. Quick now! What is the logarithm of 1000? 10? 0.1? If you answered 3, 1, -1 award yourself a major prize.

Now we come to the big leap in understanding. What is the logarithm of a number like 57? This number cannot be expressed in a nice easy format of 10 to some integer exponent. Nevertheless there is still some number that when used as the exponent will give 57. In the old days to find this number required the use of tables of logarithms. Now days we use our trusty calculators. Get your calculator and try these numbers to see for yourself. Not only will this reinforce your understanding--it will also make sure you know where all the appropriate buttons are on your calculator! Type in 57 and then find the "log" button on your calculator and press it. You should be returned the value 1.75587... This number is the logarithm of 57. If you think about it this value makes sense because we know that 57 lies between 10 and 100 and the logs of these two numbers are 1 and 2 respectively.

Finally, if we have the log of a number how do we recover the number itself? Lets continue with the example of 57. We now know that 1.75587 is the log of 57. To get from the log to the original number we must use the log value as the exponent of 10. Some calculators have a 10^x button. In this case enter 1.75587 and hit the 10^x button. Voila! You should get 57 (or something pretty close depending on how many significant digits you entered). Unfortunately not all calculators have the 10^x button. Some require that you enter 1.75587 and then hit INV and then the LOG buttons. Make sure you figure out how to use your calculator to take the log of a number and to get from the log value back to the number. You can ask me I've dealt with every manner of calculator over the semesters and I love a challenge. Try the following self test to verify your calculator savvy.