Math for Geometric Optics

• Algebra skills
  – general rules
  – some common types of equations
  – units
  – problems with several variables (substitution)

• Geometry basics

• Trigonometry
  – Pythagorean theorem
  – definitions, sine, cosine, tangent
  – angle measurement in radians
  – small angle trigonometry
What is algebra?

- Mathematics has systematic rules
  - $7 \times 6 = 6 \times 7$
    - Commutative law of multiplication
  - $3 \times (4 + 2) = 3 \times 4 + 3 \times 2$
    - Distributive law
- Symbols are used in place of numbers to emphasize that these rules hold for any numbers
  - $a \cdot (b + c) = a \cdot b + a \cdot c$
    - Note that $\times$ is not used in algebra as the multiplication symbol
- In algebra we use these rules to discover new relationships
  - $(a+b) \cdot (c+d) = a \cdot c + a \cdot d + b \cdot c + b \cdot d$
    - Repeated applications of the distributive law

Familiarity with these rules will be assumed in this class
Practical use of algebra—what ‘equals’ means

= 

• The ‘equals’ sign means that when numbers are put in for the symbols, the number on the left and right are the same
• Like a scale, if it balances and you do the same thing to both sides of the equation it still balances
  – add, subtract, divide or multiply on both sides
  – take inverse (swap numerator/denominator of the whole side)
    • Also called reciprocal
  – changes in one side that don’t change its value are also allowed
  – the object applied to each side may be in a different form on each side (substitution)

Careful, systematic application of this procedure is all that is required to do algebra successfully
Algebra helps you to transfer what you know in one field to another

\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}
\]

- The equations are of the same form and can be solved in the same way

\[
\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}
\]
**Algebra in practice-example**

- Usually you can solve for one variable for each equation you have
  - so for the equation given earlier you must be given two of \( f, s, \) and \( s' \) then you can solve for the third

\[
\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}
\]

You can also solve using the symbols, without plugging in the numbers to get new relationships such as:

\[
\frac{1}{s'} = \frac{1}{f} - \frac{1}{s} \quad \Rightarrow \quad s' = \frac{fs}{s - f}
\]
A significant digression about calculations

• In real problems the numbers you plug in have some error
  – A lens with a nominal 1000 mm focal length may have an actual focal length of 1005 mm
  – The number of digits that are accurate is called the number of significant digits

• The number of significant digits usually stays constant throughout a calculation
  – If you drop significant digits during a calculation you lose accuracy
  – Subtracting nearly equal numbers loses accuracy
Using a ruler, one student measures the distance from A to B as 103 mm. Using a very accurate laser device, a second student measures the distance from B to C as 25.872 mm. What is the distance from A to C?

Since AB is only known to the nearest mm, BC must be rounded to 26 mm. The answer, to the known accuracy, is 129 mm.

Suppose we want to divide AB and BC into three equal parts. How long would each be?

In each case, multiply the length by 1/3. The number 1/3 is an exact number. Its decimal representation can have as many significant figures as required, e.g. .3 or .33333333. The correct number must be used in any calculation.

For AB, .333*103 mm= 34mm; for BC, .33333*25.872mm=8.624mm
Simple ratio relationships

- Ratio relationships occur frequently
  - Wavelength of light, \( \lambda = \frac{c}{\nu} \)
  - Ohm’s law, \( I = \frac{V}{R} \)
  - Speed, \( V = \frac{D}{T} \)
  - Irradiance \( E = \frac{P}{A} \)

These relationships can be solved for any one of the variables by multiplying and dividing on both sides of the ‘equals’ sign.

Note that V is used for both velocity and voltage on this viewgraph. While this is inexcusable on a single viewgraph, it cannot be avoided always. The student must keep track of what symbols mean.
A memory aid for ratio relationships

\[ \frac{c}{\lambda \nu} = \frac{V}{IR} = \frac{P}{EA} \]
Units are treated just like other variables

- Example, area of a rectangle 4 cm x 5 cm
  - Area = LxW = (4cm)x(5cm) = 20 cm$^2$

- Example, amplifier efficiency 75 W out for 100 W in
  - Efficiency = (Power out)/(Power in) = 75W/100W = 0.75
    - An example of a dimensionless number

- Carrying along units has several advantages
  - warns you of mistakes
  - automatically gives answers in proper units
  - makes unit conversions easy

\[ 1 = \frac{1cm}{10mm} = \frac{1000mW}{1W} = \frac{1.602 \times 10^{-19} J}{1ev} \]

\[ 1 = 1x1 = \left( \frac{1cm}{10mm} \right) \left( \frac{1cm}{10mm} \right) = \frac{1cm^2}{100mm^2} \]

Carrying units through calculations is not required in this class. You only need to do it if you want to consistently get the right answers!
Equations with two variables-substitution

- Rules are the same, the equals sign is a balance
- Pick one equation, solve it for either one of the variables, pretending you know the other
- Substitute into the other equation and then solve as an ordinary one-variable equation
- Second variable can be found from either of the original equations

Example

\[
\begin{align*}
\text{Example} & \quad x + y = 5 \\
& \quad x - y = 1 \\
\end{align*}
\]

2nd eq gives

\[
\begin{align*}
x & = 1 + y \\
(1 + y) + y & = 5 \\
2y & = 5 - 1 = 4 \\
y & = 2 \\
\end{align*}
\]

Solve other variable

\[
\begin{align*}
x & = 1 + y = 3
\end{align*}
\]
Geometry—some basic facts

- Parallel lines
  - Don’t intersect (in Euclidean geometry)
  - Equal angles made by skew lines

- Circles
  - Terminology
    - Diameter, radius, circumference, arc
  - Formulae for circumference and area
    - $C = \pi D$, $A = \pi r^2$

- Similarity
  - Geometric figures scaled in size, zoom
  - Angles stay the same, distances all increase by the same factor, areas increase by the square of the factor
Trigonometry is the study of triangles

- A right triangle is a triangle with a right angle (90° angle)
- Trigonometric functions sine, cosine, tangent, etc are defined using right triangles
- Other triangles can be studied using a few additional properties (which we won’t need here)
Basic properties of right triangles

- Lengths of sides denoted by capital letters
- Angles denoted by small letters corresponding to opposite side
- Sum of angles is $180^\circ$
- $c=90^\circ$ so $a+b=90^\circ$, $a$ and $b$ are complementary angles
- The long side, $C$, is called the hypotenuse
- Pythagorean theorem

$$C^2 = A^2 + B^2$$
$$C = \sqrt{A^2 + B^2}$$

$3,4,5$ triangle $5^2=3^2+4^2$
Basic trigonometric functions

- \( \sin(a) = \frac{A}{C} \)
- \( \cos(a) = \frac{B}{C} \)
- \( \tan(a) = \frac{A}{B} = \frac{\sin(a)}{\cos(a)} \)
- \( \sin(b) = \frac{B}{C} \)
- \( \cos(b) = \frac{A}{C} \)
- \( \tan(b) = \frac{B}{A} = \frac{\sin(b)}{\cos(b)} = \frac{1}{\tan(a)} \)

From the Pythagorean theorem we find:

\[
\sin^2(\theta) + \cos^2(\theta) = 1
\]

This can only define the trigonometric functions for \( \Theta < 90^\circ \)
General angles

- Angles are called positive when measured in a counterclockwise direction.
- Angles are called negative when measured in a clockwise direction.
- Units for angles are either degrees or radians.
Sines and cosines and the unit circle

- The circle has a radius of 1 (unit circle)
- Measure the angle from the x axis
- For an angle less than 90°, draw the triangle as shown
  - since hypotenuse=1, cosine can be read off the x axis, sine off the y axis

For Θ>90° read the sine and cosine off the x and y axes in the same way
  - this is the definition for these angles
  - sin and cos can be positive or negative
  - tangent is still sin/cos

- Can even be defined for angles larger than 360° in this way, periodic
Measurement of angles in radians

- Length of arc swept out on the unit circle is a measure of the angle called radians
  - natural measure of angle
- $2\pi$ radians = $360^\circ$
- Your calculator can be set to work with either degrees or radians, but you must be sure of which you are using
  - to convert from degrees to radians multiply by $\pi/180 \approx 0.0175$
Trigonometric functions for small angles

• If \( b \) is a small angle
  - \( A \equiv C \)
  - \( a \equiv 90^\circ \)

• The trigonometric functions become very simple when \( b \) is measured in radians
  - \( \sin(b) = b = B/C = B/A \)
  - \( \tan(b) = b = B/A = B/C \)
  - \( \cos(b) = A/C = 1 \)

• These approximations are usually good enough when \( b < 0.1 \) radians or about 6°
Inverse trigonometric functions

- Start with example
  - triangle with A and B equal
  - \( \sin(a) = \cos(a) = \frac{1}{2}\sqrt{2}, \tan(a) = 1 \)
  - What is the angle \( a \)?
  - \( a = \sin^{-1}(\frac{1}{2}\sqrt{2}) \)

- The inverse trigonometric functions give you the angle if you know the value of the function
  - in this case if you set your calculator to degrees you find \( a = 45^\circ \), in radians \( a = 0.785 \) (which is \( \pi/4 \))
  - if you do \( \tan^{-1}(1) \) you also get \( 45^\circ \)
What to do if the last 4 viewgraphs leave you completely baffled

• You do not have to completely master trigonometry for this class

• There are just a few important concepts
  – The sine and cosine can be found for any angles
    • You can find sine, cosine and tangent using your calculator
  – The sine and cosine are periodic functions
    • We will come back to this later when we talk about the wave nature of light
  – You can convert from degrees to radians by multiplying by .175
  – For small angles, the sine, tangent and the angle itself are all the same number if the angle is given in radians
  – If you know the value of a trigonometric function and need to find the angle itself, use the inverse functions; they’re on your calculator using shift keys